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A MULTITYPE STOCHASTIC POPULATION MODEL: AN EXTENDED VERSION

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PREFACE AND SUMMARY

A previous Memorandum, RM-5407-PR, introduced and analyzed a simple multitype population model. That model was special insofar as the input process was assumed Poisson. This Memorandum analyzes a modified model that has for its input process an arbitrary renewal process. This modification allows far greater flexibility in fitting the model to some real situation than does the case of Poisson inputs.

S. C. Port is a consultant to The RAND Corporation.

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A MULTITYPE STOCHASTIC POPULATION MODEL: AN EXTENDED VERSION

1. INTRODUCTION

A previous Memorandum* introduced a multitype stochastic population model, motivated it with examples, and analyzed some of its mathematical consequences. The purpose here is to do the same with a modified model that includes the original one as a special case. This new model is the same as the previous one except that the input process is an arbitrary renewal process instead of a Poisson process. This allows far greater flexibility in fitting the model to a real situation.

The new model is as follows. A particle may be in one of r different states. Particles enter the system in bunches that arrive according to a renewal process $\eta(t)$ having waiting time $\{W_i\}$, which possesses a density and finite variance. The total number of particles in the i-th bunch is X_i. The processes $\eta(t)$ and $\{X_i\}$ are independent. The composition of the i-th bunch is $\{x_{i1}, \ldots, x_{ir}\}$; where conditional on X_i, the r-vector $\{x_{i1}, \ldots, x_{ir}\}$ is multinomial $(X_i; \alpha_1, \ldots, \alpha_r)$. After arrival into the system the particles all act independently of each other and transform their states according to a semi-Markov process having transition law $P_{ij}(t)$ and final probabilities, $\lim_{t\to\infty} P_{ij}(t) = \pi_j$. We will also assume that

$$\int_0^\infty |P_{ij}(t) - \pi_j|\, dt < \infty .$$

As before, let $M_j(t)$ and $Y_j(t)$ denote, respectively, the number of particles that enter the system in state j by time t and the number of particles in state j at time t.

For λ_1 and λ_2 negative, set

$$E\left[e^{\lambda_1 M_j(t)} e^{\lambda_2 Y_j(t)}\right] = \Phi_t(\lambda_1, \lambda_2),$$

*Sidney Port, A Multitype Stochastic Population Model, The RAND Corporation, RM-5407-PR, September 1967.

and for $o < s < 1$, set

$$E\left(s^{X_i}\right) = G(s),$$

and set $p_1(t) = \alpha_j P_{jj}(t)$, $\alpha_j[1 - P_{jj}(t)] = p_2(t)$ and $p_3(t) = \sum_{i \neq j} \alpha_i P_{ij}(t)$. Then all of the information about the $Y_j(t)$, $M_j(t)$ follows from the following integral equation that the $\Phi_t(\lambda_1, \lambda_2)$ satisfy:

$$(1.1) \quad \Phi_t(\lambda_1, \lambda_2) = P(W_1 > t) + \int_0^t G\Big[1 + \left(e^{\lambda_1 + \lambda_2} - 1\right)p_1\left(t - s\right) + \left(e^{\lambda_1} - 1\right)p_2\left(t - s\right) + \left(e^{\lambda_2} - 1\right)p_3\left(t - s\right)\Big] \Phi_{t-s}(\lambda_1, \lambda_2) dP(W_1 \leq s).$$

This equation may be derived by the usual renewal type argument and is left to the reader. We note that if the W_i are exponentially distributed with mean $1/\mu$, then the equation becomes

$$\Phi_t(\lambda_1, \lambda_2) = e^{-\mu t}\left[1 + \mu \int_0^t \widetilde{G}(s)\Phi_s e^{\mu s}\, ds\right],$$

and direct substitution shows that

$$(1.2) \qquad \Phi_\lambda(\lambda_1, \lambda_2) = \exp\left\{\mu \int_0^t [\widetilde{G}(s) - 1]\, ds\right\}.$$

Here

$$\widetilde{G}(s) = G\left[1 + \left(e^{\lambda_1 + \lambda_2} - 1\right)p_1(s) + \left(e^{\lambda_1} - 1\right)p_2(s) + \left(e^{\lambda_2} - 1\right)p_3(s)\right].$$

Setting $\lambda_1 = i\theta_1$, $\lambda_2 = i\theta_2$ in (1.2) and comparing with Eq. (2.3) of RM-5407-PR, we see that these are the same. Thus in the case of Poisson arrival, the integral Eq. (1.1) becomes the basic equation of RM-5407-PR.

We will state and prove our general results for processes in continuous time. Naturally, analogous results hold in discrete time. Unfortunately, one cannot solve Eq. (1.1) in the general case. However, we can use it to derive the behavior of the moments of $Y_j(t)$ and $M_j(t)$. Only the first two moments will be of interest to us. Set $\mu_1 = EX_1$, $\mu_2 = EX_1(X_1 - 1)$, $\gamma = (EW_1)^{-1}$. Under our assumptions, $E\eta(t) - t\gamma \to I = [\text{Var } W_1 - (EW_1)^2]\gamma^2/_2$, $t \to \infty$.

<u>Theorem 1. Under the assumptions of the model</u> $[M_j(t)/t, Y_j(t)/t]$ <u>converges in probability to</u> $(\gamma\alpha_j\mu_1, \gamma\mu_1\Pi_j)$. <u>Moreover,</u>

$$EY_j(t) = \gamma\mu_1\Pi_j t + \mu_1(\gamma J + \Pi_j I) + o(1) \quad , \tag{1.3}$$

<u>where</u>

$$J = \int_0^\infty \left[\sum_{i=1}^{r} \alpha_i P_{ij}(t) - \Pi_j\right] dt,$$

$$\text{Var } Y_j(t) = [2\mu_1^2\gamma\Pi_j^2 I + \mu_{(2)}\Pi_j^2\gamma + \mu_1\gamma\Pi_j]t + o(t), \tag{1.4}$$

$$EM_j(t) = \gamma\mu_1\alpha_j t + \mu_1\alpha_j I + o(1), \tag{1.5}$$

and

$$\text{Var } M_j(t) = [2\mu_1^2\alpha_j^2\gamma I + \mu_{(2)}\alpha_j^2\gamma + \mu_1\alpha_j\gamma]t + o(t). \tag{1.6}$$

<u>Theorem 2. Under the assumption of the model,</u>

$$\text{Cov } [M_j(t), Y_j(t)] = [2\mu_1^2\gamma\alpha_j\Pi_j I + (\mu_{(2)} + \mu_1)\alpha_j\Pi_j\gamma]t + o(t).$$

Let us now compare our general results with those for the special case when W_1 is exponential; i.e., $\eta(t)$ is Poisson with the same parameter values. Then the first term in expansion of the means is the same and the second term differs by the factor involving I. Likewise, in the asymptotic expansion of the variance, we see that there is an additional term involving I.

The exponential waiting time is a special case of the gamma. If W_1 has the gamma density,

$$f_{\gamma_1 \upsilon}(x) = \frac{\alpha^{\upsilon}}{\Gamma(\upsilon)} x^{\upsilon-1} e^{-\alpha x}, \qquad \alpha > o, \ \upsilon > o \ ,$$

then

$$EW_1 = \frac{\upsilon}{\alpha}, \ EW_1^2 = \frac{\upsilon(\upsilon + 1)}{\alpha^2} \quad ,$$

and thus

$$I = \frac{\upsilon(\upsilon + 1)}{\alpha^2} - 2\frac{\upsilon^2}{\alpha^2} = \frac{\upsilon - \upsilon^2}{\alpha^2} = \frac{\upsilon(1 - \upsilon)}{\alpha^2} \quad .$$

Thus $I > o$ if $\upsilon < 1$, $I = o$, $\upsilon = 1$, and $I < o$ if $\upsilon > 1$. The case $\upsilon = 1$ is the exponential distribution.

An interesting special case occurs when the arrival times are constant. Here it is better to formulate the model in discrete time, and it is convenient to take one time unit as the time between successive group arrivals. Then if $Y_j(n)$ is the number of particles in state j at time n, we see that

$$Y_j(n) = X_{n1} + \dots + X_{nn} \quad ,$$

where X_{ni} denotes the number of particles in state j at time n which arrive at time i. Thus

$$E\left[e^{i\theta Y_j(n)}\right] = \prod_{i=1}^{n} E\left[e^{i\theta X_{nj}}\right] .$$

Now

$$EX_{nj} = \mu_1 \sum_{k=1}^{r} \alpha_k P_{kj}^{n-i} = \mu_1 \Pi_j + o(1) \quad ,$$

and

$$\operatorname{Var} X_{ni} = (\operatorname{Var} X_1)\Big(\sum_{k=1}^{r} \alpha_k P_{kj}^{n-i}\Big)^2 + (\mu_1)\Big(\sum_{k=1}^{r} \alpha_k P_{kj}^{n-i}\Big)\Big(1 - \sum_{k=1}^{r} \alpha_k P_{kj}^{n-i}\Big)$$

$$= \mu_{(2)}\Pi_j^2 + \mu_1\Pi_j + o(1).$$

We therefore see that

$$\operatorname{Var} Y_j(n) = n[(\operatorname{Var} X_1)\Pi_j^2 + \mu_1\Pi_j(1 - \Pi_j)] + o(n).$$

Thus, computing the characteristic function, we have

$$E \exp\Big[i\theta(Y_j(n) - EY_j(n))/\sqrt{\operatorname{Var} Y_j(n)}\Big] = \prod_{i=1}^{n}\Big[1 - \frac{\theta^2}{2}\frac{\operatorname{Var} X_{ni}}{\operatorname{Var} Y_j(n)} + o(\tfrac{1}{n})\Big]$$

$$= \Big[1 - \frac{\theta^2}{2n} + o(\tfrac{1}{n})\Big]^n \to e^{-\theta^2/2}.$$

Hence we have established the following.

Theorem 3. For the case of constant group arrivals the number of partials $Y_j(n)$ in state j at time n normalized by its mean and variance is asymptotically normally distributed.

2. PROOF OF THEOREM 1

We first consider $Y_j(t)$. Setting $\lambda_1 = o$ in (1.1) and setting $E\left[e^{\lambda Y_j(t)}\right] = \Phi_t(\lambda)$, we see that

$$\Phi_t(\lambda) = 1 - F(t) + \int_o^t G\left[1 + (e^{\lambda} - 1)\, p(t - s)\right]\Phi_{t-s}(\lambda)\, dF(s),$$

where

$$p(t) = p_1(t) + p_3(t) = \sum_{i=1}^{r} \alpha_1 P_{ij}(t),$$

and $F(t) = P(W_1 \le t)$. A differentiation on λ yields

$$(2.1)\quad \Phi_t' = \int_o^t G'(\ldots)e^{\lambda} p(t - s)\Phi_{t-s}\, dF(s) + \int_o^t G(\ldots)\Phi_{t-s}'\, dF(s).$$

Let $\mu_1 = EX_1$ and $\mu_2 = EX_1(X_1 - 1)$. Setting $\lambda = o$ in (2.1) yields

$$(2.2)\quad EY_j(t) = \mu_1 \int_o^t p(t - s)\, dF(s) + \int_o^t EY_j(t - s)\, dF(s).$$

This integral equation is readily solvable by Laplace transforms. We will illustrate the technique here and henceforth leave such details to the reader. Denote Laplace transforms by ^ and let the argument of the transform function be λ. Then (2.2) yields

$$\widehat{EY}_j = \mu_1 \hat{p}\hat{F} + \widehat{EY}_j \hat{F} \quad .$$

Hence

$$\widehat{EY}_j = \mu_1 \frac{\hat{p}\hat{F}}{1 - \hat{F}} \quad .$$

Thus

$$(2.3)\qquad EY_j(t) = \mu_j \sum_{j=1}^{\infty} \int_o^t p(t - s) P(W_1 + \ldots + W_n \in ds)$$
$$= \mu_1 \int_o^t p(t - s)\, dN(s),$$

where here and in the following $N(t) = E\eta(t)$ is the renewal function. Under our assumptions on the waiting times it is known that $N(t) = \int_0^t n(s)\,ds$, where $n(s)$ is the renewal density, and that

$$\int_0^\infty |n(s) - 1/EW_1|\,ds < \infty .$$

Also

$$I = \int_0^\infty [n(s) - 1/EW_1]\,ds = \frac{EW_1^2 - 2(EW_1)^2}{(EW_1)^2} = \frac{\operatorname{Var} W_1 - (EW_1)^2}{2(EW_1)^2} .$$

Set $\gamma = (EW_1)^{-1}$. From (2.3) we see that

$$\begin{aligned} EY_j(t) - t\mu_1\gamma\Pi_j &= \mu_1 \int_0^t [p(t-s)n(s) - \gamma\Pi_j]\,ds \\ &= \mu_1 \int_0^t [p(t-s) - \Pi_j]n(s)\,ds \\ &\quad + \mu_1\Pi_j \int_0^t [n(s) - \gamma]\,ds. \end{aligned}$$

Now

$$\mu_1\Pi_j \int_0^t [n(s) - \gamma] = \mu_1\Pi_j I + o(1), \qquad t \to \infty.$$

Also

$$\begin{aligned} \int_0^t [p(t-s) - \Pi_j]n(s)\,ds &= \int_0^t [p(t-s) - \Pi_j][n(s) - \gamma]\,ds \\ &\quad + \gamma \int_0^t [p(s) - \Pi_j]\,ds. \end{aligned}$$

The first integral on the right is clearly $o(1)$ as $t \to \infty$, and the second is $\gamma J + o(1)$, where

$$J = \int_0^\infty [p(s) - \Pi_j]\,ds.$$

Hence

$$\mu_1 \int_o^t [p(t - s) - \Pi_j]n(s)\ ds = \mu_1\gamma J + o(1).$$

Thus

$$\text{(2.4)} \qquad EY_j(t) - t\mu_1\gamma\Pi_j = \mu_1\gamma J + \mu_1\Pi_j I + o(1).$$

We must now analyze the second moment $EY_j(t)^2$. Differentiating (2.1) on λ and setting $\lambda = o$ yields the equation

$$EY_j(t)^2 = 2\int_o^t \mu_1 p(t - s)EY_j(t - s)\ dF(s) + \mu_2 \int_o^t p^2(t - s)\ dF(s)$$
$$+ \mu_1 \int_o^t p(t - s)\ dF(s) + \int_o^t EY_j^2(t - s)\ dF(s)\ .$$

Solving, we find that

$$\text{(2.5)} \qquad EY_j^2(t) = 2\mu_1 \int_o^t p(t - s)EY_j(t - s)n(s)\ ds$$
$$+ \mu_2 \int_o^t p^2(t - s)n(s)\ ds + EY_j(t).$$

To establish our results we make good use of the following.

Lemma 2.1. Let $a(t)$ and $b(t)$ be nonnegative functions such that $a(t) - a_o$, $b(t) - b_o$ are in $L_1(o, \infty)$, and set $J_a = \int_o^\infty [a(t) - a]\ dt$, $J_b = \int_o^\infty [b(t) - b]\ dt$. Suppose $M(t) \geq o$, $M(t) \uparrow \infty$, $t \to \infty$, and $M(t) - tM_o \to J_M$, $t \to \infty$. Then

$$\text{(2.6)} \qquad \int_o^t b(t - s)M(t - s)a(s)\ ds - \frac{a_o b_o M_o t^2}{2}$$
$$= a_o b_o J_M t + M_o b_o J_a t + o(t).$$

Proof. We can write

$$(2.7)\qquad \int_0^t b(t-s)M(t-s)a(s)\,ds - \frac{a_o b_o M_o t^2}{2}$$

$$= \int_0^t b(t-s)[M(t-s) - (t-s)M_o]a(S)\,ds$$

$$+ M_o \int_0^t (t-s)a(s)b(t-s)\,ds - \frac{M_o a_o b_o t^2}{2}\ .$$

Now

$$b(t)[M(t) - tM_o] \sim bJ_M, \quad t \to \infty, \qquad a(t) \sim a, \quad t \to \infty\ .$$

Thus

$$(2.8)\ \int_0^t b(t-s)[M(t-s) - (t-s)M_o]a(s)\,ds \sim a_o b_o J_M t, \qquad t \to \infty.$$

On the other hand,

$$(2.9)\quad M_o \int_0^t (t-s)a(s)b(t-s)\,ds = M_o \int_0^t sb(s)a(t-s)\,ds$$

$$= M_o \int_0^t sb(s)[a(t-s) - a_o]$$

$$+ M_o a_o \int_0^t s(b(s) - b_o + \frac{M_o a_o b_o t^2}{2}\ .$$

Now integrating by parts we find that

$$(2.10)\qquad \int_0^t s[b(s) - b_o]\,ds = t\int_0^t [b(s) - b_o]\,ds$$

$$- \int_0^t ds \int_0^s [b(x) - b_o]\,dx$$

$$= tJ_b - tJ_b + o(t) = o(t).$$

Also

$$\int_0^t sb(s)[a(t-s)-a_o]\,ds = \int_0^t s[b(s)-b_o][a(t-s)-a_o]$$
$$+ b_o \int_0^t s[a(t-s)-a_o] .$$

The first term on the right is clearly $o(t)$, and the second term is

$$b_o \int_0^t s[a(t-s)-a_o]\,ds = b_o \int_0^t (t-s)[a(s)-a_o]\,ds$$
$$= tb_o \int_0^t [a(s)-a_o] - b_o \int_0^t s[a(s)-a_o]\,ds = tb_oJ_a + o(t).$$

Thus

(2.11) $$\int_0^t sb(s)[a(t-s)-a_o]\,ds = tb_oJ_a + o(t).$$

Substituting (2.10) and (2.11) into (2.9) yields

(2.12) $$M_o \int_0^t (t-s)a(s)b(t-s)\,ds = \frac{M_oa_ob_ot^2}{2} + M_ob_oJ_at + o(t) .$$

Using this and (2.8), we find from (2.7) that (2.6) holds.

We may now proceed to analyze $EY_j^2(t)$. Taking $b(t) = p(t)$, $M(t) = EY_j(t)$, and $a(t) = n(t)$ in the lemma and using (2.4) we find that

$$2\mu_1 \int_0^t p(t-s)EY_j(t-s)n(s)\,ds$$
$$= 2\mu_1\left\{\frac{\gamma^2\Pi_j^2\mu_1}{2}\,t^2 + \gamma\Pi_j(\mu_1\gamma J + \mu_1\Pi_jI)t + \mu_1\gamma\Pi_j^2It\right\} + o(t) .$$

From (2.5) we then find that

$$EY_j^2(t) = \mu_1^2\gamma^2\Pi_j^2t^2 + [2\mu_1\gamma\Pi_j(\mu_1\gamma J + \mu_1\Pi_jI) + 2\mu_1^2\gamma\Pi_j^2I]t$$
$$+ (\mu_2\Pi_j^2\gamma + \mu_1\gamma\Pi_j)t + o(t).$$

On the other hand,

$$[EY_j(t)]^2 = [EY_j(t) - \mu_1\gamma\Pi_j t][EY_u(t) - \mu_1\gamma\Pi_j t]$$

$$+ 2t\gamma\Pi_j\mu_1[EY_j(t) - \mu_1\gamma\Pi_j t] + t^2\mu_1^2\Pi_j^2\gamma^2$$

$$= \mu_1^2\Pi_j^2\gamma^2 t^2 + 2\gamma\Pi_j\mu_1[\mu_1\gamma J + \mu_1\Pi_j I]t + o(t).$$

Thus

$$\text{(2.13)} \qquad \text{Var } Y_j(t) = EY_j^2(t) - [EY_j(t)]^2$$

$$= [2\mu_1^2\gamma\Pi_j^2 I + \mu_2\Pi_j^2\gamma + \mu_1\gamma\Pi_j]t + o(t).$$

Turning our attention now to $M_j(t)$ we may proceed in a similar manner. Setting $\lambda_2 = o$ in (1.1) and letting $\Phi_t(\lambda) = E\left[e^{\lambda M_j(t)}\right]$ we find that

$$\Phi_t(\lambda) = 1 - F(t) + \int_o^t G[1 + (e^\lambda - 1)\alpha_j]\Phi_{t-s}(\lambda)\, dF(s).$$

Proceeding as before we find that

$$EM_j(t) = \mu_1\alpha_j N(t)$$

and

$$EM_j(t)^2 = 2\mu_1\alpha_j \int_o^t EM_j(t - s)n(s)\, ds + \mu_2\alpha_j^2 N(t) + EM_j(t)$$

$$= 2\mu_1^2\alpha_j^2 \int_o^t N(t - s)n(s)\, ds + \mu_2\alpha_j^2 N(t) + \mu_1\alpha_j N(t).$$

In Lemma 2.1 set $b(t) = 1$, $M(t) = N(t)$, and $a(s) = n(s)$.

Then

$$\int_o^t N(t - s)n(s)\, ds = \frac{\gamma^2 t^2}{2} + 2\gamma I t + o(t).$$

Thus

$$EM_j^2(t) = 2\mu_1^2\alpha_j^2\left[\frac{\gamma^2t^2}{2} + 2\gamma It\right] + \mu_2\alpha_j^2\gamma t + \mu\alpha_j\gamma t + o(t),$$

since

$$[EM_j(t)]^2 = \mu_1^2\alpha_j^2 N(t)^2 = \mu_1^2\alpha_j^2\{[N(t) - \gamma t]^2 + 2\gamma t[N(t) - \gamma t] + \gamma^2t^2\}$$

$$2\mu_1^2\alpha_j^2\gamma I + \mu_1^2\alpha_j^2\gamma^2t^2 + o(1).$$

Thus

$$\text{(2.14)} \qquad \operatorname{Var} M_j(t) = [2\mu_1^2\alpha_j^2\gamma I + \mu_2\alpha_j^2\gamma + \mu\alpha_j\gamma]t + o(t).$$

We may now easily establish the law of large numbers for $Y_j(t)$ and $M_j(t)$. Indeed, by (2.13) we know that $\operatorname{Var} Y_j(t) = O(t)$, and thus

$$P\left[\left|\frac{Y_j(t)}{t} - \gamma\Pi_j\mu_1\right| > \epsilon\right] \le \frac{\operatorname{Var} Y_j(t)}{\epsilon^2t^2} = O(\tfrac{1}{t}) .$$

Likewise (2.14) shows that

$$P\left[\left|\frac{M_j(t)}{t} - \gamma\mu_1\alpha_j\right| > \epsilon\right] \le \frac{\operatorname{Var} \mu_j(t)}{\epsilon^2t^2} = o(\tfrac{1}{t}) .$$

This completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

Computing

$$\left[\frac{\partial^2 \phi_t}{\partial\lambda_1 \partial\lambda_2}\right]_{\lambda_1=\lambda_2=o},$$

we find that

$$(3.1) \quad \mathrm{Cov}\,(M_j(t),\, Y_j(t)) = \int_o^t \{\mu_{(2)}(p_1(s) = p_3(s))(p_1(s) + p_2(s))$$

$$+ \mu p_1(s) + \mu(p_1(s) + p_2(s))\; EY_j(t - s)$$

$$+ \mu_1(p_1(s) = p_3(s))EM_j(t - s)\}n(s)\; ds.$$

Consider the third term on the right. Since $p_1 + p_2 = \alpha_j$, this term becomes

$$\mu\alpha_j \int_o^t EY_j(t - s)n(s)\; ds.$$

In Lemma 2.1 set $b(t) = 1$, $a(s) = n(s)$, and $M(t) = EY_j(t)$. Then in view of Eq. (1.3)

$$(3.2) \quad \mu\alpha_j\left\{\int_o^t EY_j(t - s)\; n(s)\; ds - \frac{\gamma^2\mu_1\Pi_j t^2}{2}\right\}$$

$$= \alpha_j\gamma\mu_1^2[\gamma J + \Pi_j I]t\; \alpha_j\mu_1^2\gamma\Pi_j It + o(t).$$

Similarly, in the fourth term on the right, $p_1(s) + p_3(s) = \sum_{i=1}^{r} \gamma_i P_{ij}(s)$, and thus this term is

$$\mu_1 \int_o^t P_{\alpha j}(s)EM_j(t - s)n(s)\; ds.$$

In Lemma 2.1 set $b(t) = P_{\alpha j}(t)$, $EM_j(t) = M(t)$, and $a(t) = n(t)$. Then in view of Eq. (1.5) we see that

$$\mu_1 \int_0^t P_{\alpha j}(s) EM_j(t - s) n(s)\, ds - \frac{\mu_1^2 \Pi_j \gamma^2 \alpha_j t^2}{2} \tag{3.3}$$

$$= 2\gamma \Pi_j \mu_1^2 I t + o(t).$$

Now

$$EY_j(t) EM_j(t) - t^2 \mu_1^2 \gamma^2 \alpha_j \Pi_j$$

$$= \mu_1^2 \alpha_j \gamma [\gamma J + \Pi_j I] t + \mu_1^2 \gamma \Pi_j \alpha_j I t + o(t).$$

Thus from (3.1) - (3.3) we see that

$$\text{Cov}\ (Y_j(t),\ M_j(t)) = [2\mu_1^2 \alpha_j \gamma \Pi_j I + (\mu_{(2)} + \mu_1) \alpha_j \Pi_j \gamma] t + o(t).$$

This establishes Theorem 2.

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10. ABSTRACT

Introduction and analysis of an extended version of the multitype stochastic population model considered in RM-5407-PR. The new model, which is the same as the previous one except that the input process is an arbitrary renewal process instead of a Poisson process, includes the previous model as a special case. General results are stated and proved for processes in continuous time. Analogous results hold, of course, in discrete time. The main advantage of the input process modification is that it allows far greater flexibility in fitting the model to a real situation, although no attempt to do so is made in the present study.

11. KEY WORDS

Statistical methods and processes
Models
Inventory control
Maintenance
Personnel
Environmental health and safety